Operational Calculus (Applied to RC and RL Circuits)

Defining py as $\frac{dy}{dt}$ and $\frac{1}{p}[y]$ as $\int_{-\infty}^{t} y\, dt'$ where y is a function of t we can write the differential equation $\frac{dy}{dt} + ay = x$ as $py + ay = x$.

Algebraically this can be written as $y(p + a) = x$, giving $y = \frac{1}{p+a}x$ where $\frac{1}{p+a}$ can be thought of as an operator acting on the function x to give the function y as an output.

Taking $p[e^{at}y]$ we have,

$$p[e^{at}y] = ae^{at}y + e^{at}py = e^{at}(p + a)$$

Returning to the expression $y(p + a) = x$ we can multiply both sides of this equation by e^{at} to give

$$e^{at}y(p + a) = e^{at}x$$

Therefore

$$p[e^{at}y] = e^{at}x$$

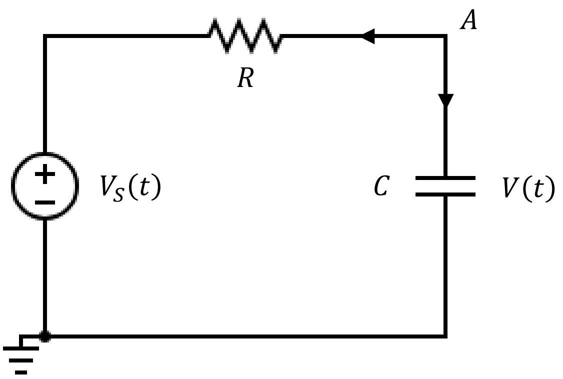
Given that p and $\frac{1}{p}$ are inverse operations

$$e^{at}y = \frac{1}{p}[e^{at}x]$$

Which gives y as

$$y = e^{-at}\frac{1}{p}[e^{at}x]$$

<u>RC Circuit</u>



The algebraic sum of currents meeting at node A is zero, the current passing through the resister being $\frac{V(t)-V_S(t)}{R}$ while the current passing through the capacitor is $C\frac{d}{dt}V(t)$.

Therefore,

$$\frac{V - V_S}{R} + C\frac{dV}{dt} = 0$$

Dividing both sides by C and moving $\frac{1}{RC}V_S$ to the other side of the equation we have

$$\frac{dV}{dt}+\frac{1}{RC}V=\frac{1}{RC}V_S$$

This is now in the form of the differential equation previously encountered and solved, $py+ay=x$. Where y is V, a is $\frac{1}{RC}$ and x is $\frac{1}{RC}V_S$.

In order to model the response, $V(t)$ of the above circuit when a constant voltage V_m is applied at $t=0$ we can let $V_S=u(t)V_m$ where $u(t)$ is the unit step function.

To model the response once a constant voltage of V_m is removed from the circuit when $t=0$ we can let $V_s=\big(1-u(t)\big)V_m$.

$$V=e^{-\frac{t}{RC}}\frac{1}{p}\left[e^{\frac{t}{RC}}\frac{1}{RC}u(t)V_m\right]=e^{-\frac{t}{RC}}\int_{-\infty}^{t}e^{\frac{t\prime}{RC}}\frac{1}{RC}u(t')V_m\,dt\,'$$

Given the function being integrated has a value of zero when $t<0$ we can integrate from 0 to t and bring $u(t)$ out of the integral. Bringing out the other constants $\frac{1}{RC}$ and V_m we have

$$V=u(t)\frac{V_m}{RC}e^{-\frac{t}{RC}}\int_0^t e^{\frac{t\prime}{RC}}\,dt'=u(t)\frac{V_m}{RC}e^{-\frac{t}{RC}}\left[RCe^{\frac{t\prime}{RC}}\right]_0^t=u(t)\frac{V_m}{RC}e^{-\frac{t}{RC}}\left[RCe^{\frac{t}{RC}}-RC\right]$$
$$=u(t)V_me^{-\frac{t}{RC}}\left[e^{\frac{t}{RC}}-1\right]=u(t)V_m\left[1-e^{-\frac{t}{RC}}\right]$$

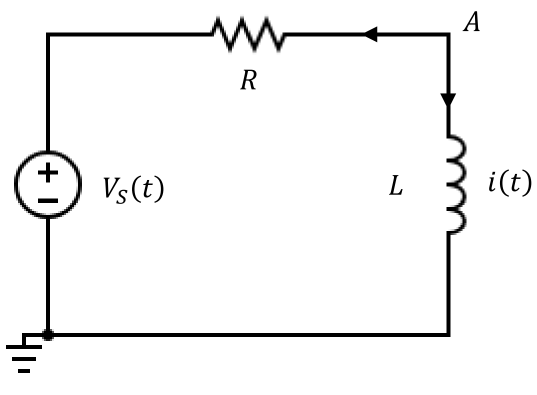
Letting $V_S=\big(1-u(t)\big)V_m$ we have,

$$V=e^{-\frac{t}{RC}}\frac{1}{p}\left[e^{\frac{t}{RC}}\frac{1}{RC}\big(1-u(t)\big)V_m\right]=e^{-\frac{t}{RC}}\int_{-\infty}^{t}e^{\frac{t\prime}{RC}}\frac{1}{RC}\big(1-u(t')\big)V_m\,dt'$$

Now the function being integrated has a value of zero when $t>0$, therefore it becomes pointless to integrate past zero. Additionally, because we are only interested in what happens after $t=0$ we can decide to integrate from $-\infty$ to 0. $\big(1-u(t)\big)$ has a value of one when t is between $-\infty$ and 0, therefore when integrating from $-\infty$ to 0, $\big(1-u(t)\big)$ can be removed from the integral. Taking out the constants $\frac{1}{RC}$ and V_m we have

$$V=\frac{V_m}{RC}e^{-\frac{t}{RC}}\int_{-\infty}^{0}e^{\frac{t\prime}{RC}}\,dt'=\frac{V_m}{RC}e^{-\frac{t}{RC}}\left[RCe^{\frac{t\prime}{RC}}\right]_{-\infty}^{0}=\frac{V_m}{RC}e^{-\frac{t}{RC}}[RC]=V_me^{-\frac{t}{RC}}$$

<u>RL Circuit</u>



Again, the algebraic sum of currents meeting at node A is zero. The current passing through the inductor being $i(t)$, the current passing through the inductor can be found as $\frac{L}{R}\frac{d}{dt}i(t) - \frac{1}{R}V_S(t)$ given that the voltage across the inductor is $L\frac{di}{dt}$.

We have

$$\frac{L}{R}\frac{di}{dt} - \frac{1}{R}V_S + i = 0$$

Multiplying by R and dividing by L, as well as moving $\frac{1}{L}V_S$ to the other side of the equation we have

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}V_S$$

This being written in the form $py + ay = x$ where y is i, a is $\frac{R}{L}$ and x is $\frac{1}{L}V_S$.

To model the currents response in the circuit to the application and removal of a constant voltage V_m at the point $t = 0$, we can use $V_S = u(t)V_m$ for application and $V_S = \big(1 - u(t)\big)V_m$ for removal.

$$\begin{aligned} i = e^{-\frac{Rt}{L}}\frac{1}{p}\left[e^{\frac{Rt}{L}}\frac{1}{L}u(t)V_m\right] = e^{-\frac{Rt}{L}}\int_{-\infty}^{t} e^{\frac{Rt\prime}{L}}\frac{1}{L}u(t')V_m\,dt' = u(t)\frac{V_m}{L}e^{-\frac{Rt}{L}}\int_0^t e^{\frac{Rt\prime}{L}}\,dt' \\ = u(t)\frac{V_m}{L}e^{-\frac{Rt}{L}}\left[\frac{L}{R}e^{\frac{Rt\prime}{L}}\right]_0^t = u(t)\frac{V_m}{L}e^{-\frac{Rt}{L}}\left[\frac{L}{R}e^{\frac{Rt}{L}} - \frac{L}{R}\right] = u(t)\frac{V_m}{R}e^{-\frac{Rt}{L}}\left[e^{\frac{Rt}{L}} - 1\right] \\ = u(t)\frac{V_m}{R}\left[1 - e^{-\frac{Rt}{L}}\right] \end{aligned}$$

In this circuit the inductor resists change in the current once the voltage V_m is applied, as t approaches infinity the current $i(t)$ approaches its maximum values I_{Max} which is $\frac{V_m}{R}$, therefore $i(t)$ can be written as

$$i = u(t)I_{Max}\left[1 - e^{-\frac{Rt}{L}}\right]$$

Letting $V_S = \big(1 - u(t)\big)V_m$

$$\begin{aligned} i = e^{-\frac{Rt}{L}}\frac{1}{p}\left[e^{\frac{Rt}{L}}\frac{1}{L}\big(1 - u(t)\big)V_m\right] = e^{-\frac{Rt}{L}}\int_{-\infty}^{t} e^{\frac{Rt\prime}{L}}\frac{1}{L}\big(1 - u(t')\big)V_m\,dt' = \frac{V_m}{L}e^{-\frac{Rt}{L}}\int_{-\infty}^{0} e^{\frac{Rt\prime}{L}}\,dt' \\ = \frac{V_m}{L}e^{-\frac{Rt}{L}}\left[\frac{L}{R}e^{\frac{Rt}{L}}\right]_{-\infty}^{0} = \frac{V_m}{L}e^{-\frac{Rt}{L}}\left[\frac{L}{R}\right] = \frac{V_m}{R}e^{-\frac{Rt}{L}} \end{aligned}$$

Given that $I_{Max} = \frac{V_m}{R}$

$$i = I_{Max}e^{-\frac{Rt}{L}}$$